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Graphic Transformation Method applied to Asymmetric Neural Networks

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1. Introduction

Understanding the nature of a non-equilibrium state has been one of the most important problems in physics, chemistry, and biology. A steady state is a good starting point in understanding the nature of a non-equilibrium state. The steady state shares some common properties with the equilibrium state, such as the invariance under a translation in time. In neural networks with asymmetric synaptic connections, the steady state can also be treated as a generalization of the equilibrium state of a neural network with symmetric connections.

In this analysis, I present an improved version of the graphic transformation method (Mogi 1993, 1994) with which we can express the steady state as a generalization of the equilibrium. In the graphic transformation method, one expresses the distribution weight of a state in terms of the product of the transition rate constants over the spanning in-trees with the state in question as the sink. This method is developed on top of the graphic method of King & Altman (1956). The asymmetric factors which contribute to the deviation from the equilibrium are obtained explicitly as a function of the asymmetric synaptic weights of the neural network.

2. The Original Graphic Transformation Method

The system I consider in this paper is a discrete steady state system. The system is composed of *N* discrete states *(S(1), S(2), S(3), ..., S(N)),* which are connected through kinetic transitions. The transitions between the states are described as a Markow process. A typical example of such a system would be a biochemical reaction network with *N* quasi-stable states, or the dynamics of the firing states in neural networks.

In the argument that follows, I develop the mathematical formalisms in a way that is applicable to transitions between discrete states in general. It is immediately applicable to asymmetric neural networks.

The transitions between the states are described by the Master equation

$$\frac{\partial \rho(i,t)}{\partial t} = \sum_{j=1}^{N} K(i,j)\rho(j,t) - K(j,i)\rho(i,t) \qquad (2.1)$$

where $\rho(i,t)$ denotes the state density of the state *S(i)* at time *t*. In the case of a neural network with N neurons, the index *i* represents a state in $\{0,1\}^N$.

We assume that the rate constants satisfy the relations

$$\frac{K(i,j)}{K(j,i)} = e^{-\beta(E(i)-E(j)+d(i,j))} \qquad (2.2)$$

where *E(i)* and *E(j)* are the energy values for *S(i)* and *S(j)*, respectively, and *d(i,j)* is the "asymmetric term". The parameter β is the inverse of *kT*, where *k* is Boltzmann's constant ($k = 1.38 \times 10^{-23} J\ K^{-1}$), and T is the absolute temperature. When *d(i,j)=0*, there is an equilibrium solution of equation (2.1) which satisfies the requirement of detailed balancing (Bridgman 1928).

$$\rho(i,t) = \frac{e^{-\beta E(i)}}{\sum_{i=1}^{N} e^{-\beta E(i)}} \qquad (2.3)$$

The equilibrium solution for symmetric neural nets is represented by (2.3), where the equilibrium energy can be given as

$$E(S) = -\frac{1}{2}\sum_{i=1}^{N}\sum_{j=1}^{N} w_{ij}s_i s_j + \sum_{i=1}^{N} s_i\theta_i \qquad (2.4)$$

Now let us consider the asymmetric case.

From the definition of the asymmetric terms, we have

$$d(i,j) = -d(j,i) \qquad (2.5)$$

For neural nets, the asymmetric terms can be given as (Mogi 1994)

$$d(S^p,S^q) = -\frac{1}{4}\sum_i \sum_j \left(w_{ij} - w_{ji}\right)\left(s_i^p - s_i^q\right)\left(s_j^p + s_j^q\right) \qquad (2.6)$$

where w_{ij} is the synaptic weight from the *j*th neuron to the *i*th neuron.

In the following discussion, we consider only the steady state. So we drop the time *t* from the variables.

Normalization condition can now be written as

$$\sum_{i=1}^{N} \rho(i) = 1 \qquad (2.7)$$

The balance equation for the steady state can be written as

$$\sum_{j=1}^{N} K(i,j)\rho(j) - K(j,i)\rho(i) = 0 \qquad (2.8)$$

As is well known, the solution for equation (2.8) is given by the Cramer's formula.

$$\rho(i) = \frac{W(i)}{\sum_{i=1}^{N} W(i)}$$

where

$$W(i) = \begin{vmatrix} \sum_j K(1,j) & -K(2,1) & \cdots & 0 & \cdots & -K(N,1) \\ -K(1,2) & \sum_j K(2,j) & \cdots & 0 & \cdots & -K(N,2) \\ \vdots & \vdots & \ddots & \vdots & & \vdots \\ 1 & 1 & \cdots & 1 & \cdots & 1 \\ \vdots & \vdots & & \vdots & \ddots & \vdots \\ -K(1,N) & -K(2,N) & \cdots & 0 & \cdots & \sum_j K(N,j) \end{vmatrix} \qquad (2.9)$$

where the *i*th column takes the value of *0* (except for the crossing with the *i*th row), and the *i*th row takes the value of *1*.

Alternatively, the solution for equation (2.8) can be represented by the graphic method initiated by King & Altman (1956). For a review of the King & Altman method, see, for example, Chou (1989).

In the King & Altman method, we consider the set of all possible spanning in-trees with the *N* states *S(1)...S(N)* as the vertices. We express the spanning in-trees which have the state *S(i)* as the sink as $g_i(m)$ ($m=1,2,3...n_g$, n_g being the same for all *i*). We write the underlying spanning tree of $g_i(m)$ (the spanning tree obtained by removing the direction from the edges of $g_i(m)$) as $g(m)$. We write the directed edge from state *S(j)* to state *S(i)* as $e_d(i,j)$, and the undirected edge between *S(i)* and *S(j)* as $e(i,j)=e(j,i)$.

The number of spanning in-trees n_g is given as (Cayley 1878)

$$n_g = N^{N-2} \qquad (2.10)$$

The weight *W(i)* for the state *S(i)* can then be written as

$$W(i) = \sum_m \prod_{e_d(p,q) \in g_i(m)} K(p,q) \qquad (2.11)$$

where

$$\prod_{e_d(p,q) \in g_i(m)} K(p,q) \qquad (2.12)$$

denotes the product of the rate constants corresponding to the directed edges of the spanning in-tree $g_i(m)$.

The graphic transformation method acts on the form (2.11), and makes it possible to express the weight for state distribution as a generalization of the Boltzmann distribution (2.3). Namely, it "separates" the contribution of the asymmetric term *d(i,j)*.

The graphic transformation is defined as the transformation of a spanning in-tree $g_i(m)$ into another spanning in-tree $g_j(m)$ (Fig.1 (a)).

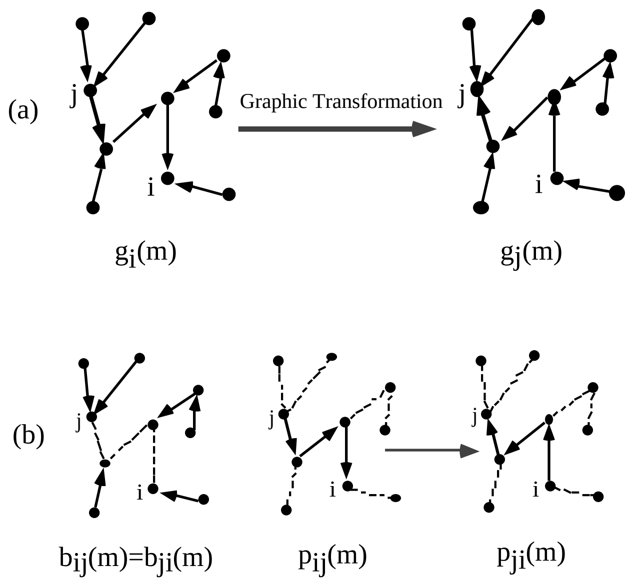


Fig.1 The Graphic Transformation Method

We define the subgraph $p_{ij}(m)$ of $g_i(m)$ as the path starting from the *j*th state and ending at the *i*th state. We denote the subgraph of $g_i(m)$ obtained by subtracting $p_{ij}(m)$ from $g_i(m)$ as $b_{ij}(m)$ (Fig.1(b)). Note that $b_{ij}(m)$ is symmetric in its suffices. Namely,

$$g_i(m) - p_{ij}(m) = b_{ij}(m) = b_{ji}(m) = g_j(m) - p_{ji}(m) \qquad (2.13)$$

The graphic transformation from $g_i(m)$ to $g_j(m)$

can then be accomplished by the reversal of the directed edges belonging to $p_{ij}(m)$ to obtain $p_{ji}(m)$. Symbolically, the graphic transformation can be written as

$$g_i(m) = b_{ij}(m) + p_{ij}(m) \xrightarrow{\text{graphic transformation}} b_{ji}(m) + p_{ji}(m) = g_j(m) \quad (2.14)$$

Now we study how the term

$$\prod_{e_d(p,q)\in g_i(m)} K(p,q) \quad (2.15)$$

in the expression of weight (2.11) is transformed in the process of the graphic transformation. Namely, we are interested in how the term (2.15), which is expressed in terms of the spanning in-tree $g_i(m)$, is expressed in terms of spanning in-tree $g_j(m)$. We find that the transformation of this term is

$$\begin{aligned}
&\prod_{e_d(p,q)\in g_i(m)} K(p,q) \\
&= \prod_{e_d(p,q)\in g_i(m)-p_{ij}(m)} K(p,q) \prod_{e_d(p,q)\in p_{ij}(m)} K(p,q) \\
&= \prod_{e_d(p,q)\in g_i(m)-p_{ij}(m)} K(p,q) \prod_{e_d(p,q)\in p_{ij}(m)} K(q,p)e^{-\beta(E(p)-E(q)+d(p,q))} \\
&= \prod_{e_d(p,q)\in g(m)-p_{ij}(m)} K(p,q)\times \\
&\prod_{e_d(p,q)\in p_{ji}(m)} K(p,q) \prod_{e_d(p,q)\in p_{ij}(m)} e^{-\beta(E(p)-E(q)+d(p,q))} \\
&= e^{-\beta(E(i)-E(j))} \prod_{e_d(p,q)\in g_j(m)} K(p,q)\; e^{-\beta\sum_{e_d(p,q)\in p_{ij}(m)} d(p,q)}
\end{aligned} \quad (2.16)$$

Note that in the process of the graphic transformation, the factors

$$\prod_{e_d(p,q)\in p_{ij}(m)} e^{-\beta(E(p)-E(q))} \quad (2.17)$$

cancel out except for the terms for the both ends of the path $p_{ij}(m)$, resulting in the term

$$e^{-\beta(E(i)-E(j))} \quad (2.18)$$

This is the essential feature of the graphic transformation method, and we can take advantage of this feature to obtain the weight expression explicitly as a function of the asymmetric terms. Namely, if we take a particular state $S(i_0)$ as the standard state, we find that the weight for the state S(i) can be written as

$$W(i) = e^{-\beta E(i)} \frac{\sum_m e^{-\beta\sum_{e_d(p,q)\in p_{ii_0}(m)} d(p,q)} \prod_{e_d(p,q)\in g_{i_0}(m)} K(p,q)}{\sum_m \prod_{e_d(p,q)\in g_{i_0}(m)} K(p,q)} \quad (2.19)$$

Note in the symmetric case $w_{ij} = w_{ji}$ the weight (2.19) reduces to

$$W(i) = e^{-\beta E(i)} \quad (2.20)$$

which gives the equilibrium distribution of (2.3), as expected.

3. New Formalism of Graphic Transformation Method

In the previous section I described the original form of the graphic transformation method, which succeeds in expressing the contribution of the asymmetric term explicitly. As a result, the steady state weight distribution has been expressed as a generalization of the equilibrium distribution.

One shortcoming of the above formalism is that it contains the standard state $S(i_0)$ explicitly. Although the actual result does not depend on the choice of the standard state, it would be advantageous if we could obtain a formalism which does not require an explicit choice of the standard state. In this section, I develop a new formalism of the graphic transformation method which satisfies such a criterion.

The new formalism starts with the expression of the rate constants as

$$K(i,j) = L(i,j)e^{\frac{-\beta(E(i)-E(j)+d(i,j))}{2}} \quad (3.1)$$

where

$$L(i,j) = L(j,i) = \frac{K(i,j) + K(j,i)}{e^{\frac{-\beta(E(i)-E(j)+d(i,j))}{2}} + e^{\frac{-\beta(E(j)-E(i)+d(j,i))}{2}}} \quad (3.2)$$

Note that under this definition, the rate constants *K(i,j)* satisfy the Arrenius relations (2.2).

The terms *L(i,j)* can be interpreted to represent the strength of connection between the states, which is common for the transition in both directions.

The weight expression (2.11) is rewritten as

$$
\begin{aligned}
W(i) &= \sum_{m} \prod_{e_d(p,q) \in g_i(m)} K(p,q) \\
&= \sum_{m} \prod_{e_d(p,q) \in g_i(m)} L(p,q) e^{\frac{-\beta\left(E(p)-E(q)+d(p,q)\right)}{2}} \\
&= \sum_{m} \prod_{e(p,q) \in g(m)} L(p,q) \prod_{e_d(p,q) \in g_i(m)} e^{\frac{-\beta\left(E(p)-E(q)\right)}{2}} \prod_{e_d(p,q) \in g_i(m)} e^{\frac{-\beta d(p,q)}{2}}
\end{aligned}
\tag{3.3}
$$

Now let us examine how the terms in (3.3) are transformed in the process of the graphic transformation from $g_i(m)$ to $g_j(m)$.

The first term in (3.3), namely

$$
\prod_{e(p,q) \in g(m)} L(p,q) \tag{3.4}
$$

is common for all states S(i), so it remains invariant during the graphic transformation.

The second term in (3.3) is transformed as

$$
\begin{aligned}
&\prod_{e_d(p,q) \in g_i(m)} e^{\frac{-\beta\left(E(p)-E(q)\right)}{2}} \\
&= \prod_{e_d(p,q) \in g_i(m)-p_{ij}(m)} e^{\frac{-\beta\left(E(p)-E(q)\right)}{2}} \prod_{e_d(p,q) \in p_{ij}(m)} e^{\frac{-\beta\left(E(p)-E(q)\right)}{2}} \\
&= \prod_{e_d(p,q) \in g_j(m)-p_{ji}(m)} e^{\frac{-\beta\left(E(p)-E(q)\right)}{2}} \prod_{e_d(p,q) \in p_{ji}(m)} e^{\frac{-\beta\left(E(p)-E(q)\right)}{2}} \times \\
&\quad \prod_{e_d(p,q) \in p_{ij}(m)} e^{-\beta\left(E(p)-E(q)\right)} \\
&= \prod_{e_d(p,q) \in g_j(m)} e^{\frac{-\beta\left(E(p)-E(q)\right)}{2}} \prod_{e_d(p,q) \in p_{ij}(m)} e^{-\beta\left(E(p)-E(q)\right)} \\
&= e^{-\beta\left(E(i)-E(j)\right)} \prod_{e_d(p,q) \in g_j(m)} e^{\frac{-\beta\left(E(p)-E(q)\right)}{2}}
\end{aligned}
\tag{3.5}
$$

Therefore, we have

$$
\prod_{e_d(p,q) \in g_i(m)} e^{\frac{-\beta\left(E(p)-E(q)\right)}{2}} = e^{-\beta E(i)} \gamma(m) \tag{3.6}
$$

where $\gamma(m)$ is a quantity independent of the index *i*. It can be shown that

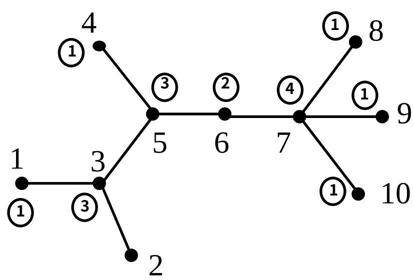
$$
\gamma(m) = e^{-\beta \sum_i \frac{1}{2} \lambda_i(m) E(i)} \tag{3.7}
$$

where $\lambda_i(m)$ is the number of edges incident with the state *S(i)* in the spanning tree *g(m)*.

We have

$$
\sum_i \lambda_i(m) = 2(N-1) \tag{3.8}
$$

As an illustration of the intuitive meaning of γ(m), we give γ(m) for the example of Fig.2.



$$
\gamma(m) = e^{-\frac{\beta}{2}\left(E(1)+E(2)+3E(3)+E(4)+3E(5)+4E(7)+E(8)+E(9)+E(10)\right)}
$$

Fig.2 Illustration of the term γ(m)

The third term in (3.3) is transformed as

$$
\begin{aligned}
&\prod_{e_d(p,q) \in g_i(m)} e^{\frac{-\beta d(p,q)}{2}} \\
&= \prod_{e_d(p,q) \in g_i(m)-p_{ij}(m)} e^{\frac{-\beta d(p,q)}{2}} \prod_{e_d(p,q) \in p_{ij}(m)} e^{\frac{-\beta d(p,q)}{2}} \\
&= \prod_{e_d(p,q) \in g_i(m)-p_{ij}(m)} e^{\frac{-\beta d(p,q)}{2}} \prod_{e_d(p,q) \in p_{ji}(m)} e^{\frac{-\beta d(p,q)}{2}} \prod_{e_d(p,q) \in p_{ij}(m)} e^{\beta d(p,q)} \\
&= \prod_{e_d(p,q) \in g_i(m)} e^{\frac{-\beta d(p,q)}{2}} \prod_{e_d(p,q) \in p_{ij}(m)} e^{\beta d(p,q)}
\end{aligned}
\tag{3.9}
$$

In this case, the transformation coefficient

$$
\prod_{e_d(p,q) \in p_{ij}(m)} e^{\beta d(p,q)} \tag{3.10}
$$

contains both the indices *i* and *j* and cannot be expressed as a product of two factors containing

only the indices i and j, respectively. Therefore, we cannot simplify the third term in (3.3) any further.
Putting (3.6) into (3.3), we obtain

$$W(i) = e^{-\beta E(i)} \sum_m \gamma(m) \prod_{e(p,q)\ g(m)} L(p,q) e^{-\beta \sum_{e_d(p,q)\ g_i(m)} \frac{d(p,q)}{2}} \tag{3.11}$$

or, in a "normalized form" as

$$W(i) = e^{-\beta E(i)} \frac{\sum_m \gamma(m) \prod_{e(p,q)\ g(m)} L(p,q) e^{-\beta \sum_{e_d(p,q)\ g_i(m)} \frac{d(p,q)}{2}}}{\sum_m \gamma(m) \prod_{e(p,q)\ g(m)} L(p,q)} \tag{3.12}$$

Equation (3.12) expresses the distribution of the states as a generalization of the Boltzmann distribution. It is easily seen that the distribution reduces to the Boltzmann distribution when the asymmetric terms *d(p,q)* are made zero.

The intuitive meaning of the expression (3.12) can be understood as follows. In the symmetric case, there is no difficulty in assigning the contributions of the energy values *E(i)* to the distribution of the states. We just assign a factor of

$$e^{-\beta E(i)} \tag{3.13}$$

to the weight of the state *S(i)*. The fact we can do this comes from the fact that in the symmetric case, the weight transforms, under the graphic transformation, as

$$W(i) = e^{-\beta(E(i)-E(j))} W(j) \tag{3.14}$$

so that the factor (3.13) can be locally assigned to each state.

On the other hand, when there are non-zero asymmetric terms, the weight transforms under the graphic transformation as

$$W(i) = e^{-\beta(E(i)-E(j))} \prod_{e_d(p,q)\ p_{ij}(m)} e^{\beta d(p,q)} W(j) \tag{3.15}$$

In this case, there is no apparent way of assigning the contributions of the asymmetric term "locally". The new formalism of the graphic transformation method provides such a method, by assigning to each state *S(i)* a contribution of

$$\prod_{e_d(p,q)\ g_i(m)} e^{\frac{-\beta d(p,q)}{2}} \tag{3.16}$$

taken over the spanning in-tree $g_i(m)$. Note that by this arrangement, when we consider the ratio of the weights of the states *S(i)* and *S(j)*, the transformation coefficient of (3.15) is recovered (Fig.3).

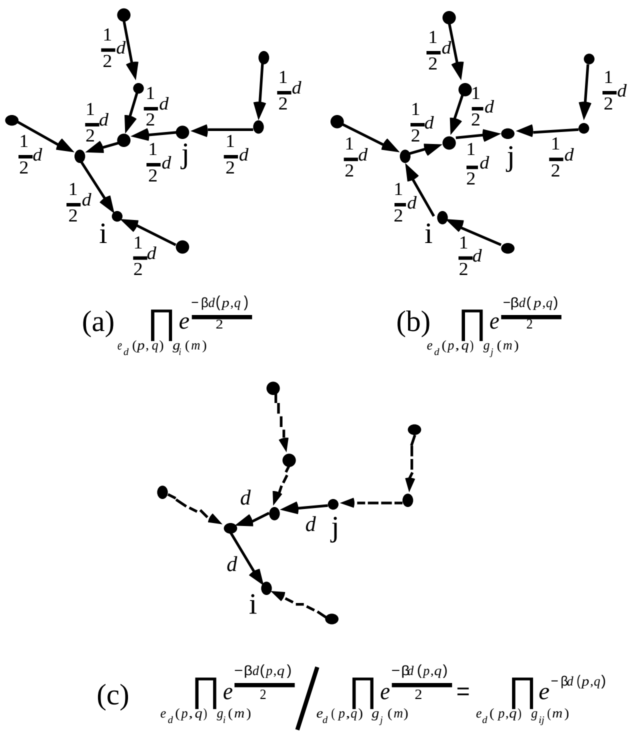


Fig.3 Assignment of the asymmetric terms

To summarize, with the new formalism of the graphic transformation method, we have succeeded in assigning the contribution of the asymmetric terms locally to each state, in a way which is consistent with the transformation property of the weights as is prescribed in (3.15).

4. Steady Flow in the System

We now investigate the steady flow in the system.

When the connections between the states are symmetric, there is no net steady flow between the states, and the detailed balancing (Bridgman 1928) is satisfied. When there is an asymmetric term, the steady flow is in general not zero. Therefore, the largeness of the steady flow can be regarded as a measure of deviation from the equilibrium.

The steady flow from the *y*th state to the *x*th state is defined as

$$F(x,y) = K(x,y)\rho(y) - K(y,x)\rho(x) \tag{4.1}$$

Using the results of previous section, we have

$$
\begin{aligned}
&F(x,y)\\
&= L(x,y)\times\\
&\left(\rho(y)e^{\frac{-\beta(E(x)-E(y)+d(x,y))}{2}}-\rho(x)e^{\frac{-\beta(E(y)-E(x)+d(y,x))}{2}}\right)\\
&= L(x,y)\times\\
&\left(\frac{w(y)}{W}e^{\frac{-\beta(E(x)-E(y)+d(x,y))}{2}}-\frac{w(x)}{W}e^{\frac{-\beta(E(y)-E(x)+d(y,x))}{2}}\right)\\
&= L(x,y)\frac{e^{\frac{-\beta(E(x)+E(y))}{2}}}{W}\sum_m \gamma(m)\prod_{e(p,q)\in g(m)}L(p,q)\times\\
&\left(e^{-\beta\left(\sum_{e_d(p,q)\in g_y(m)}\frac{d(p,q)}{2}+\frac{d(x,y)}{2}\right)}-e^{-\beta\left(\sum_{e_d(p,q)\in g_x(m)}\frac{d(p,q)}{2}+\frac{d(y,x)}{2}\right)}\right)
\end{aligned}
\tag{4.2}
$$

where

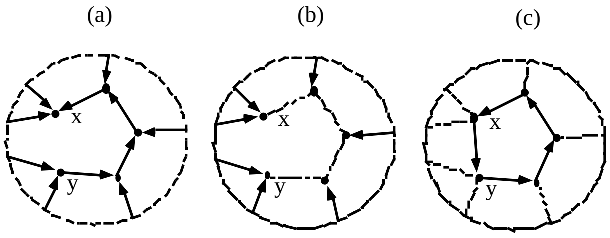
$$
W\equiv\sum_m \gamma(m)\prod_{e(p,q)\in g(m)}L(p,q) \tag{4.3}
$$

Now we have

$$
\begin{aligned}
&e^{-\beta\left(\sum_{e_d(p,q)\in g_y(m)}\frac{d(p,q)}{2}+\frac{d(x,y)}{2}\right)}-e^{-\beta\left(\sum_{e_d(p,q)\in g_x(m)}\frac{d(p,q)}{2}+\frac{d(y,x)}{2}\right)}\\
&= e^{-\beta\left(\sum_{e_d(p,q)\in b_{xy}(m)}\frac{d(p,q)}{2}\right)}\times\\
&\left(e^{-\beta\left(\sum_{e_d(p,q)\in p_{yx}(m)}\frac{d(p,q)}{2}+\frac{d(x,y)}{2}\right)}-e^{-\beta\left(\sum_{e_d(p,q)\in p_{xy}(m)}\frac{d(p,q)}{2}+\frac{d(y,x)}{2}\right)}\right)\\
&= e^{-\beta\left(\sum_{e_d(p,q)\in b_{xy}(m)}\frac{d(p,q)}{2}\right)}\times\\
&\left(e^{-\beta\left(\sum_{e_d(p,q)\in \Lambda(y,x,m)}\frac{d(p,q)}{2}\right)}-e^{-\beta\left(\sum_{e_d(p,q)\in \Lambda(x,y,m)}\frac{d(p,q)}{2}\right)}\right)
\end{aligned}
\tag{4.4}
$$

where we recall that $b_{xy}(m)$ $(=b_{yx}(m))$ is the common subgraph of $g_x(m)$ and $g_y(m)$ defined as in (2.13). The directed graph $\Lambda(x,y,m)$ is defined as the loop obtained by adding the directed edge $e_d(y,x)$ to the path $P_{xy}(m)$ (See Fig.4).

$$
\Lambda(x,y,m)\equiv p_{xy}(m)+e_d(y,x) \tag{4.5}
$$



$g_x(m)$ $\quad$ $b_{xy}(m)=b_{yx}(m)$ $= g_x(m)-p_{xy}(m)$ $\quad$ $\Lambda_{xy}(m)$ $= p_{xy}(m)+e_d(y,x)$

Fig. 4 Definition of the loop $\Lambda_{xy}(m)$

Steady flow is finally obtained as

$$
\begin{aligned}
&F(x,y)\\
&= L(x,y)\frac{e^{\frac{-\beta(E(x)+E(y))}{2}}}{W}\times\\
&\sum_m \gamma(m)\prod_{e(p,q)\in g(m)}L(p,q)e^{-\beta\left(\sum_{e_d(p,q)\in b_{xy}(m)}\frac{d(p,q)}{2}\right)}\times\\
&\left(e^{-\beta\left(\sum_{e_d(p,q)\in \Lambda(y,x,m)}\frac{d(p,q)}{2}\right)}-e^{-\beta\left(\sum_{e_d(p,q)\in \Lambda(x,y,m)}\frac{d(p,q)}{2}\right)}\right)\\
&= 2L(x,y)\frac{e^{\frac{-\beta(E(x)+E(y))}{2}}}{W}\times\\
&\sum_m \gamma(m)\prod_{e(p,q)\in g(m)}L(p,q)e^{-\beta\left(\sum_{e_d(p,q)\in b_{xy}(m)}\frac{d(p,q)}{2}\right)}\times\\
&\sinh\left(\beta\sum_{e_d(p,q)\in \Lambda(x,y,m)}\frac{d(p,q)}{2}\right)
\end{aligned}
\tag{4.6}
$$

Note that the steady flow is zero when the asymmetric terms are made to be zero, as expected.

5. Multiple Energy Values

One interpretation of the weight distribution of (3.11) is that the steady state in a neural net with asymmetric connections is characterized by multiple energy values, whereas the equilibrium state in neural net with symmetric connections is characterized by a single energy value. We can rewrite (3.11) as

$$W(i) = \sum_m \gamma(m) \prod_{e(p,q) \in g(m)} L(p,q) e^{-\beta \left[E(i) + \sum_{e_d(p,q) \in g_i(m)} \frac{d(p,q)}{2} \right]}$$

(5.1)

Here, there are multiple energy values *E(i, m)* corresponding to each spanning in-tree $g_i(m)$

$$E(i,m) = E(i) + \sum_{e_d(p,q) \in g_i(m)} \frac{d(p,q)}{2}$$

(5.2)

The relative importance of the energy value E(i,m) is given by the term

$$\sum_m \gamma(m) \prod_{e(p,q) \in g(m)} L(p,q)$$

(5.3)

6. Conclusion

In this paper, I have applied the graphic transformation method to the analysis of discrete steady state system. The application to asymmetric neural networks is almost automatic, which boils down to the assignment of the asymmetric term *d* as a function of the synaptic weights..

The graph theory has been applied to several aspects of biological systems, including the biochemical reaction network (e.g., Chou et al. (1979), Chou (1990), Mazur (1990), Mazur & Kuchinski (1992), Mitchell et al. (1989), Volkenstein & Goldstein (1966), Zou & Deng (1984)). The graphic transformation method I discussed in this paper belong to this kind of effort to understand the biological systems mathematically.

I have described the new formalism of the graphic transformation method, which has the advantage of not having to rely on a standard state for its exposition. Also, the significance of the asymmetric terms is clearer in the new formalism.

There are some interesting natures of the steady state revealed by the graphic transformation method. The argument in section 3 shows that there is no way of locally assigning the energy value. The energy value for a particular state can only expressed in terms of global property of the reaction network;namely by the product of the asymmetric term over the spanning in-tree, which in fact involves *all* the states in the network. Therefore, we cannot assign a energy value locally to a state. Such a property of the steady state as compared to the equilibrium is expected to be important not only for discrete systems discussed in this paper, but also for steady state systems in general.

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